STOKES ELEMENTS ON CUBIC MESHES YIELDING DIVERGENCE–FREE APPROXIMATIONS

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Abstract. Conforming piecewise polynomial spaces with respect to cubic meshes are constructed for the Stokes problem in arbitrary dimensions yielding exactly divergence–free velocity approximations. The derivation of the finite element pair is motivated by a smooth de Rham complex that is well–suited for the Stokes problem. We derive the stability and convergence properties of the new elements as well as the construction of reduced elements with less global unknowns.

1. Introduction

This article constructs two families of stable and conforming finite element pairs for the Stokes problem with respect to cubic meshes in arbitrary dimension. In particular, we shall construct finite element spaces $\boldsymbol{X}_h \subset \boldsymbol{H}^1(\Omega)$ and $Y_h \subset L^2(\Omega)$ satisfying the discrete inf–sup (or Ladyzenskaja–Babuska–Brezzi) condition

$$\beta\|q\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}\in\boldsymbol{X}_h\setminus\{0\}} \frac{\int_\Omega (\operatorname{div}\boldsymbol{v})q\,dx}{\|\boldsymbol{v}\|_{H^1(\Omega)}}, \tag{1.1}$$

and, in addition, satisfy the conforming property

$$\boldsymbol{Z}_h := \{\boldsymbol{v}\in\boldsymbol{X}_h : \int_\Omega (\operatorname{div}\boldsymbol{v})q\,dx = 0,\ \forall q\in Y_h\} \subset \boldsymbol{Z} := \{\boldsymbol{w}\in\boldsymbol{H}^1(\Omega) :\ \operatorname{div}\boldsymbol{w}\equiv 0\}. \tag{1.2}$$

Condition (1.2) states that the discretely divergence–free subspace is in fact divergence–free pointwise. As a result, the computed velocity approximation of the Stokes problem is exactly solenoidal.

It is well–known (cf. [5]) that the discrete Stokes problem based on the velocity–pressure formulation is well–posed if and only if the discrete inf–sup condition (1.1) is satisfied. It implies that the divergence operator acting on the velocity space $\boldsymbol{X}_h$ has a surjective–type property with a bounded right inverse; in short, the discrete inf–sup condition implies the inclusion $Y_h \subseteq P_h \operatorname{div}\boldsymbol{X}_h$, where $\operatorname{div}\boldsymbol{X}_h$ is the image $\boldsymbol{X}_h$ under the divergence operator and P_h is the L^2–projection onto Y_h. If this condition is satisfied, then the discrete velocity approximation satisfies the quasi–optimal estimate

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)} \le C\Big[\inf_{\boldsymbol{v}\in\boldsymbol{X}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{H^1(\Omega)} + \nu^{-1}\inf_{q\in Y_h}\|p-q\|_{L^2(\Omega)}\Big], \tag{1.3}$$

where $C > 0$ is a constant with scaling β^{-1} and ν denotes the viscosity of the fluid. Numerical experiments in, e.g., [16] show that the scaling in (1.3) is sharp, and therefore the error may deteriorate for small viscosity–values.

The conforming property (1.2) on the other hand implies the reverse relation $\operatorname{div}\boldsymbol{X}_h \subseteq Y_h$, and thus, finite element pairs satisfying both conditions (1.1)–(1.2) satisfy the equality $\operatorname{div}\boldsymbol{X}_h = Y_h$. In this setting, the velocity approximation satisfies the decoupled and ν–independent error estimate

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} \le C\inf_{\boldsymbol{v}\in\boldsymbol{X}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{H^1(\Omega)},$$

where again, $C > 0$ scales like β^{-1}. Thus, finite element pairs satisfying both conditions (1.1)–(1.2) have enhanced stability properties and are robust with the problem's parameters.

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The construction of our finite element pairs is motivated by a smooth de Rham complex (or Stokes complex [11]). In two dimensions, this complex is given by the sequence

$$\mathbb{R} \longrightarrow H^2(\Omega) \overset{\text{curl}}{\longrightarrow} \boldsymbol{H}^1(\Omega) \overset{\text{div}}{\longrightarrow} L^2(\Omega) \longrightarrow 0. \tag{1.4}$$

If the domain is simply connected, then this complex is exact, i.e., the range of each map is the kernel of the succeeding map. The exactness property implies that the divergence operator is surjective from $\boldsymbol{H}^1(\Omega)$ onto $L^2(\Omega)$. Along with an estimate of the right–inverse, this result implies the inf-sup condition in the continuous setting. In n–dimensions we view functions in $\boldsymbol{H}^1(\Omega)$ and $L^2(\Omega)$ as $(n-1)$–forms and n–forms, respectively, via their proxies. In particular, if $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})^t$ and $q \in L^2(\Omega)$, then we make the identifications

$$\boldsymbol{v} \sim \sum_{j=1}^{n} v^{(j)} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots dx^n, \quad q \sim q dx^1 \wedge \cdots dx^n.$$

Let d denote the exterior derivative and set $H\Lambda^\ell(\Omega)$ to be the space of L^2 ℓ–forms with exterior derivatives in L^2. Then the n–dimensional de Rham complex with minimal L^2 smoothness is [1, 2]

$$\mathbb{R} \longrightarrow H\Lambda^0(\Omega) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} H\Lambda^{n-2}(\Omega) \overset{d}{\longrightarrow} H\Lambda^{n-1}(\Omega) \overset{d}{\longrightarrow} H\Lambda^n(\Omega) \longrightarrow 0.$$

The Stokes complex is obtained by simply imposing additional regularity in the second–to–last and third–to–last spaces is the sequence:

$$\mathbb{R} \longrightarrow H\Lambda^0(\Omega) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \hat{H}\Lambda^{n-2}(\Omega) \overset{d}{\longrightarrow} H^1\Lambda^{n-1}(\Omega) \overset{d}{\longrightarrow} H\Lambda^n(\Omega) \longrightarrow 0, \tag{1.5}$$

where $H^1\Lambda^{n-1}(\Omega)$ denotes the space of $(n-1)$–forms with coefficients in $H^1(\Omega)$, and $\hat{H}\Lambda^{n-2}(\Omega) := \{\omega \in H\Lambda^{n-2}(\Omega) : d\omega \in H^1\Lambda^{n-1}(\Omega)\}$. This complex is the guiding tool to develop stable finite element Stokes pairs that yield divergence–free velocity approximations. Namely, starting with a $H\Lambda^0(\Omega)$–conforming finite element space, we follow the sequence (1.5) to deduce properties of the finite element pair $\boldsymbol{X}_h \times Y_h$.

The development of conforming finite element pairs yielding divergence–free approximations was initiated by Scott and Vogelius in [21]. Here, the authors showed that the pair $\boldsymbol{\mathcal{P}}_k - \mathcal{P}_{k-1}^{dc}$ is stable in two dimensions on simplicial triangulations provided the polynomial degree satisfies $k \geq 4$ and if the triangulation does not contain singular vertices. These results have since been expanded in [14, 11]. Similar to the simplicial case, the construction of Stokes pairs yielding divergence–free approximations on Cartesian meshes is mostly limited to the two dimensional case [4, 24, 15]. A noticeable exception is [8, 7, 9, 10], where the authors developed stable spaces yielding divergence–free approximations in two and three dimensions within an isogeometric framework. In terms of global regularity, the finite element spaces developed in this paper are in between the $\boldsymbol{H}(\text{div}\,;\Omega)$–conforming Nedelec finite element spaces [18] and these isogeometric spaces. Due to the differences between our elements and those given in [8, 7], and due to a lack of a Fortin operator, new tools are developed to prove the necessary inf–sup condition. In particular we first derive a local inf–sup condition with imposed boundary conditions and then translate this result to the global level by exploiting the element's degrees of freedom.

The rest of the paper is organized as follows. In Section 2 we provide the notation that is used throughout the paper and state some preliminary results. We then state the local velocity and pressure spaces and their degrees of freedom in Section 3. In addition, we derive some local characterizations of the divergence operator acting on polynomial spaces. In Section 4 we define the global finite element spaces and derive the desired inf-sup condition. In Section 5 we construct finite element pairs with similar approximation and stability properties, but with less unknowns. Finally, we apply these elements to the Stokes problem in Section 6 and prove error estimates in the energy norm.

2. **Preliminaries**

2.1. **Notation.** Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary parallel to the coordinate axes. We denote by $\mathcal{T}_h$ a conforming triangulation of Ω consisting of cubical elements $\{Q\}_{Q\in\mathcal{T}_h}$ such that the boundary of each element is parallel to the coordinate axes. For an element $Q \in \mathcal{T}_h$, we denote by h_Q its diameter and by $\square_s(Q)$ the set of faces of Q with dimension s. In particular, $\square_s(Q)$ denotes the set of faces where $(n-s)$ coordinates are one of two constant values. Consequently, the cardinality of this set is $|\square_s(Q)| = 2^{n-s}\binom{n}{s}$. We denote by $\square_s^{(i)}(Q)$ the set of s–dimensional faces of Q where x_i is constant and by $\hat{\square}_s^{(i)}(Q)$ the set of s–dimensional faces of Q where x_i is not constant; the cardinality of these sets are $|\square_s^{(i)}(Q)| = 2^{n-s}\binom{n-1}{s}$ and $|\hat{\square}_s^{(i)}(Q)| = 2^{n-s}\binom{n-1}{s-1}$, respectively, for any i. The analogous global sets with respect to $\mathcal{T}_h$ are denoted by $\square_s$, $\square_s^{(i)}$ and $\hat{\square}_s^{(i)}$.

We denote by $\mathcal{P}_{\vec{k}}(D) := \mathcal{P}_{k_1,k_2,\dots,k_n}(D)$, the space of polynomials on $D \subset \mathbb{R}^n$ of degree at most k_i in x_i, and set $\mathcal{Q}_k(D) := \mathcal{P}_{\vec{k}}(D)$ with $k_i = k$ for all i. We further denote the vector–valued space

$$\mathbf{Q}_k^-(Q) = \{\boldsymbol{v} \in (\mathcal{Q}_k(Q))^n :\ v^{(i)} \in \mathcal{P}_{\vec{k}}(Q) \text{ with } k_i = k \text{ and } k_j = k-1 \text{ for } i \neq j\}.$$

For example $\mathbf{Q}_k^-(Q) = \mathcal{P}_{k,k-1}(Q)\times\mathcal{P}_{k-1,k}(Q)$ and $\mathbf{Q}_k^-(Q) = \mathcal{P}_{k,k-1,k-1}(Q)\times\mathcal{P}_{k-1,k,k-1}\times\mathcal{P}_{k-1,k-1,k}(Q)$ in two and three dimensions, respectively (cf. [18]). The dimensions of these spaces are $\dim \mathcal{Q}_k(Q) = (k+1)^n$ and $\dim \mathbf{Q}_k^-(Q) = n(k+1)k^{n-1}$.

Lemma 2.1 ([3])**.** *A function $q \in \mathcal{Q}_k(Q)$ is uniquely determined by the values*

$$\int_S q\kappa \qquad \kappa \in \mathcal{Q}_{k-2}(S), \quad S \in \square_s(Q),\ s = 0,1,\dots,n, \tag{2.1}$$

where $\int_S q$ with $S \in \square_0(Q)$ is understood to be the evaluation of q at the vertex S.

For an element $Q \in \mathcal{T}_h$ with face $S \in \square_s(Q)$ $(1 \le s \le n)$, we denote by b_S the bubble function with respect to S. In particular, $b_S \in \mathcal{Q}_2(Q)$ is a quadratic polynomial in each variable that vanishes on ∂S and takes the value one at the center of S. If $S \in \square_n(Q) = \{Q\}$, then we denote the bubble function by b_Q. We remark that $\nabla b_S \not\equiv 0$ on ∂S; however, the gradient of the bubble function vanishes on $(s-2)$–dimensional sub–faces of S.

3. **The Local Stokes Elements**

In this section we define the local velocity and pressure finite elements for the Stokes problem. In particular, we define the local spaces of these elements and the unisolvent sets of degrees of freedom. In addition, we derive a characterization of the divergence acting on the local velocity space, crucial for the stability analysis of the global spaces defined in the subsequent section.

3.1. **Local velocity finite element spaces on cubic meshes in $\mathbb{R}^n$.** In this section we define the local velocity space and degrees of freedom. First, we require the following technical result.

Lemma 3.1. *Suppose $\boldsymbol{v} = (v^{(1)}, v^{(2)}, \dots, v^{(n)}) \in \mathbf{Q}_3^-(Q)$ satisfies*

$$\int_S v^{(i)} = 0,\ \int_S \frac{\partial v^{(i)}}{\partial x_i} = 0 \qquad S \in \square_s^{(i)}(Q),\ s = 0,1,\dots,m, \tag{3.1a}$$

for all $1 \le i \le n$ and for some $0 \le m \le n-2$. Then $v = 0$, $\partial v^{(i)}/\partial x_i = 0$ on $\square_s(Q)$ for all $1 \le i \le n$ and $0 \le s \le m$. If in addition,

$$\int_S v^{(i)} = 0 \qquad S \in \square_{m+1}^{(i)}(Q), \tag{3.1b}$$

then $v = 0$ on $\square_{m+1}(Q)$.

Proof. The proof is by induction on m. The case $m=0$ is clearly true since $\square_0^{(i)}(Q)=\square_0(Q)$ for all $i=1,2,\ldots,n$.

Assume that $\boldsymbol{v}$ and $\frac{\partial v^{(i)}}{\partial x_i}$ vanish on all $S\in\square_m(Q)$ for some $0\le m\le n-3$ and all $1\le i\le n$. Let $S\in\square_{m+1}(Q)$. By the induction hypothesis, we have $\boldsymbol{v}=0$ and $\frac{\partial v^{(i)}}{\partial x_i}=0$ on ∂S for all $1\le i\le n$. If x_j is constant on S, i.e., $S\in\square_{m+1}^{(j)}(Q)$, then $v^{(j)}|_S,\partial v^{(j)}/\partial x_j|_S\in\mathcal{Q}_2(S)$. Therefore we may write $v^{(j)}|_S=b_Sq^{(j)}$ and $\partial v^{(j)}/\partial x_j|_S=b_Sp^{(j)}$ for some $q^{(j)},p^{(j)}\in\mathbb{R}$. By (3.1a) we conclude that both $v^{(j)}$ and $\partial v^{(j)}/\partial x_j$ vanish on S.

If $S\in\hat{\square}_{m+1}^{(j)}(Q)$ then there exists exactly two m–dimensional faces $S^{(1)},S^{(2)}\subset\partial S$ such that x_j is constant on $S^{(1)}$ and $S^{(2)}$. Denote by ∇_S and $\nabla_{S^{(i)}}$ the surface gradient of S and $S^{(i)}$, respectively. Then, with the correct orientation, we have $\nabla_S(\cdot)=(\nabla_{S^{(i)}}(\cdot),\frac{\partial(\cdot)}{\partial x_j})$. Therefore, since $v^{(j)}$ and $\partial v^{(j)}/\partial x_j$ vanish on $S^{(1)}$ and $S^{(2)}$, we conclude that $\nabla_S v^{(j)}=0$ on $S^{(1)}$, $S^{(2)}$. Since $v^{(j)}|_S\in\mathcal{Q}_3(S)$, and vanishes on ∂S, we may write $\bar{v}^{(j)}|_S=b_Sq$ for some $q\in\mathcal{Q}_1(S)$. Consequently,

$$0=\nabla_S v^{(j)}|_{S^{(i)}}=q\nabla_S b_S|_{S^{(i)}}. \tag{3.2}$$

We then conclude from (3.2) that $q=0$ on $S^{(i)}$, and therefore, since $q\in\mathcal{Q}_1(S)$, $q\equiv 0$. Thus $v^{(j)}=0$ and $\partial v^{(j)}/\partial x_j=0$ on S. The first assertion of the lemma now follows from induction. The second assertion follows from the exact same arguments. $\square$

The local velocity space and degrees of freedom are given in the next lemma.

Lemma 3.2. *Any function $\boldsymbol{v}\in\boldsymbol{\mathcal{Q}}_3^-(Q)$ is uniquely determined by the values (cf. Figure 1)*

$$\int_S v^{(i)},\ \int_S\frac{\partial v^{(i)}}{\partial x_i}\qquad S\in\square_s^{(i)}(Q),\ s=0,1,\ldots,n-2, \tag{3.3a}$$

$$\int_S v^{(i)}\qquad S\in\square_{n-1}^{(i)}(Q), \tag{3.3b}$$

$$\int_Q \boldsymbol{v}\cdot\boldsymbol{\kappa}\qquad \boldsymbol{\kappa}\in\boldsymbol{\mathcal{Q}}_1^-(Q) \tag{3.3c}$$

for $i=1,2,\ldots,n$.

Proof. The number of degrees of freedom given in (3.3) equals

$$\begin{aligned}2n\sum_{s=0}^{n-2}|\square_s^{(i)}(Q)|+n|\square_{n-1}^{(i)}(Q)|+\dim\boldsymbol{\mathcal{Q}}_1^-(Q)&=2n\sum_{s=0}^{n-2}2^{n-s}\binom{n-1}{s}+2n+2n\\&=4n\Big[-1+\sum_{s=0}^{n-1}2^{n-1-s}\binom{n-1}{s}\Big]+4n\\&=4n3^{n-1}=\boldsymbol{\mathcal{Q}}_3^-(Q).\end{aligned}$$

by the binomial theorem. Thus, to show that (3.3) form a unisolvent set over $\boldsymbol{\mathcal{Q}}_3^-(Q)$, it suffices to show that a function $\boldsymbol{v}\in\boldsymbol{\mathcal{Q}}_3^-(Q)$ vanishes on (3.3) if and only if $\boldsymbol{v}\equiv 0$.

If $\boldsymbol{v}$ vanishes on (3.3), then $\partial v^{(i)}/\partial x_i=0$ on $\square_{n-2}(Q)$ and $\boldsymbol{v}=0$ on $\square_{n-1}(Q)$ by Lemma 3.1. Therefore $\boldsymbol{v}=b_Q\boldsymbol{q}$ for some $\boldsymbol{q}\in\boldsymbol{\mathcal{Q}}_1^-(Q)$. Finally, the last set of degrees of freedom (3.3c) implies $\boldsymbol{v}\equiv 0$. $\square$

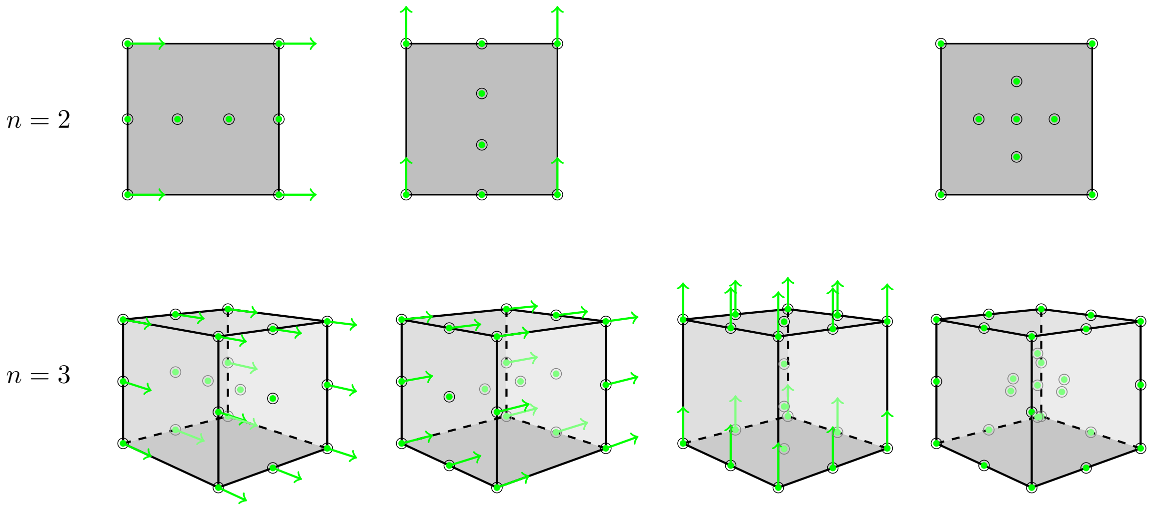

FIGURE 1. Degrees of freedom of the velocity (left) and pressure (right) elements in two and three dimensions.

Remark 3.3. For $S \in \square_s(Q)$ let $\{\boldsymbol{n}_S^{(j)}\}_{j=1}^{n-s}$ be an orthonormal set of vectors orthogonal to the tangent space of S. We may then write the degrees of freedom (3.3a)–(3.3b) as

$$\int_S \boldsymbol{v}\cdot\boldsymbol{n}_S^{(j)},\ \int_S \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_S^{(j)}}\cdot\boldsymbol{n}_S^{(j)} \qquad S\in\square_s(Q),\ s=0,1,\dots,n-2,\ j=1,2,\dots,n-s, \tag{3.4a}$$

$$\int_S \boldsymbol{v}\cdot\boldsymbol{n}_S \qquad S\in\square_{n-1}(Q). \tag{3.4b}$$

We shall use the (equivalent) degrees of freedom (3.4), (3.3c) in Section 4 below.

3.2. **Local pressure finite element spaces on cubic meshes in $\mathbb{R}^n$.** The local pressure space consists of tensor–product quadratic polynomials, namely $\mathcal{Q}_2(Q)$. By Lemma 2.1 any function $q\in\mathcal{Q}_2(Q)$ is uniquely determined by the values

$$\int_S q, \quad S\in\square_s(Q),\ s=0,1,\dots,n. \tag{3.5}$$

Consequently, the subspace

$$\breve{\mathcal{Q}}_2(Q) := \{q\in\mathcal{Q}_2(Q):\ q \text{ vanishes on all } (n-2)\text{–dimensional faces of } Q\},$$

has dimension $2n+1$. Namely, a function $q\in\breve{\mathcal{Q}}_2(Q)$ is uniquely determined by its average over each $(n-1)$–dimensional face, and its average over Q. This conversation also leads to the following result:

Lemma 3.4. *Any function $q\in\mathcal{Q}_2(Q)$ is uniquely determined by the values*

$$\int_S q \qquad S\in\square_s(Q),\ s=0,1,\dots,n-2,$$
$$\int_Q q\kappa \qquad \kappa\in\breve{\mathcal{Q}}_2(Q).$$

3.3. **Local characterizations of the divergence.**

Lemma 3.5. *Let $\boldsymbol{v}\in\boldsymbol{\mathcal{Q}}_3^-(Q)$. Suppose that* $\operatorname{div}\boldsymbol{v}=0$ *and that $\boldsymbol{v}$ vanishes on the boundary of Q. Then $\boldsymbol{v}\equiv 0$.*

Proof. Let $\bar{\boldsymbol{v}} \in \Lambda^{n-1}(Q)$ denote the $(n-1)$–form with vector proxy $\boldsymbol{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(n)})$; i.e.,

$$\bar{\boldsymbol{v}} = \sum_{i=1}^{n} v^{(i)} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n,$$

where the hat indicates a suppressed argument. For a vector–valued function $\boldsymbol{\kappa}$, we also denote by $\underline{\boldsymbol{\kappa}} \in \Lambda^1(Q)$ the one–form given by

$$\underline{\boldsymbol{\kappa}} = \sum_{i=1}^{n} \kappa^{(i)} dx^i.$$

The divergence–free condition on $\boldsymbol{v}$ is equivalent to $d\bar{\boldsymbol{v}} = 0$, where d denotes the exterior derivative. Moreover the boundary condition $\boldsymbol{v}|_{\partial Q} = 0$ implies that the trace of $\bar{\boldsymbol{v}}$ vanishes on ∂Q. Therefore there exists $\boldsymbol{\varphi} \in \mathring{H}\Lambda^{n-2}(Q)$ such that $\bar{\boldsymbol{v}} = d\boldsymbol{\varphi}$ [1, 2]. Here, $\mathring{H}\Lambda^{n-2}(Q)$ denotes the space of $L^2(Q)$ $(n-2)$–forms with exterior derivative in $L^2(Q)$ and vanishing trace. By Stokes Theorem, we have for any $\boldsymbol{\kappa} \in \mathbf{\Omega}_1^-(Q)$

$$\int_Q \boldsymbol{v} \cdot \boldsymbol{\kappa} = \int_Q \bar{\boldsymbol{v}} \wedge \underline{\boldsymbol{\kappa}} = \int_Q d\boldsymbol{\varphi} \wedge \underline{\boldsymbol{\kappa}} = (-1)^{n-1} \int_Q \boldsymbol{\varphi} \wedge d\underline{\boldsymbol{\kappa}} = 0. \tag{3.6}$$

The last equality is due to the identity $d\underline{\boldsymbol{\kappa}} = 0$ for $\boldsymbol{\kappa} \in \mathbf{\Omega}_1^-(Q)$. Indeed, we have

$$d\underline{\boldsymbol{\kappa}} = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial \kappa^{(j)}}{\partial x_k} dx^k \wedge dx^j = \sum_{j=1}^{n} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\partial \kappa^{(j)}}{\partial x_k} dx^k \wedge dx^j = 0$$

since $\kappa^{(j)}$ is constant in x_k for $j \neq k$. By (3.6) and Lemma 3.2 we conclude that $\boldsymbol{v} \equiv 0$. □

Remark 3.6. An alternative proof of Lemma 3.5 without the direct use of differential forms is given as follows. If $\boldsymbol{v} \in \mathbf{\Omega}_3^-(Q) \cap \boldsymbol{H}_0^1(Q)$ then $\boldsymbol{v} = b_Q \boldsymbol{\kappa}$ for some $\boldsymbol{\kappa} \in \mathbf{\Omega}_1^-(Q)$. If $\operatorname{div} \boldsymbol{v} = 0$, then by the chain rule, $\nabla b_Q \cdot \boldsymbol{\kappa} + b_Q \operatorname{div} \boldsymbol{\kappa} = 0$. Restricting this identity to the boundary of Q, we conclude that $\boldsymbol{\kappa} \cdot \nabla b_Q|_{\partial Q} = 0$. In particular $\boldsymbol{\kappa} \cdot \nabla b_Q|_S = 0$ for all $S \in \square_{n-1}(Q)$. However, a simple calculation shows $\nabla b_Q|_S = a_S \boldsymbol{n}_S b_S$ for some non–zero constant a_S. Therefore $\boldsymbol{\kappa} \cdot \boldsymbol{n}|_{\partial Q} = 0$. Due to this identity and the inclusion $\boldsymbol{\kappa} \in \mathbf{\Omega}_1^-(Q)$ we conclude from [3, pg. 9] that $\boldsymbol{\kappa} \equiv 0$.

Theorem 3.7. *Define the spaces*

$$\mathring{\mathbf{\Omega}}_3^-(Q) : = \mathbf{\Omega}_3^-(Q) \cap \boldsymbol{H}_0^1(Q), \qquad \mathring{\Omega}_2(Q) := \check{\Omega}_2(Q) \cap L_0^2(Q).$$

Then the divergence operator is a bijection from $\mathring{\mathbf{\Omega}}_3^-(Q)$ to $\mathring{\Omega}_2(Q)$. Moreover, there holds $\|\boldsymbol{v}\|_{H^1(Q)} \le C\|\operatorname{div} \boldsymbol{v}\|_{L^2(Q)}$ for all $\boldsymbol{v} \in \mathring{\mathbf{\Omega}}_3^-(Q)$ with $C > 0$ independent of $\boldsymbol{v}$ and the size of Q.

Proof. By the definitions of $\mathring{\mathbf{\Omega}}_3^-(Q)$ and $\mathring{\Omega}_2(Q)$ and Stokes Theorem, we see that $\operatorname{div} \mathring{\mathbf{\Omega}}_3^-(Q) \subseteq \mathring{\Omega}_2(Q)$. Therefore, in light of Lemma 3.5, it suffices to show that $\dim\big(\operatorname{div} \mathring{\mathbf{\Omega}}_3^-(Q)\big) = \dim \mathring{\Omega}_2(Q)$ to prove the bijective property.

By Lemmas 3.2 and 3.5 we have

$$\dim\big(\operatorname{div} \mathring{\mathbf{\Omega}}_3^-(Q)\big) = \dim \mathring{\mathbf{\Omega}}_3^-(Q) = \dim \mathbf{\Omega}_1^-(Q) = 2n.$$

On the other hand, we have by Lemma 3.4,

$$\dim \mathring{\Omega}_2(Q) = \dim \check{\Omega}_2(Q) - 1 = 2n.$$

Therefore $\dim\big(\operatorname{div} \mathring{\mathbf{\Omega}}_3^-(Q)\big) = \dim \mathring{\Omega}_2(Q)$, and therefore $\operatorname{div} : \mathring{\mathbf{\Omega}}_3^-(Q) \to \mathring{\Omega}_2(Q)$ is a bijection.

Finally, let $F : \hat{Q} \to Q$ be an affine mapping, where $\hat{Q} = (0,1)^n$ is the reference element. In particular, $F(\hat{x}) = B\hat{x} + b$, where $b \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$ is a diagonal matrix with entries proportional to h_Q. Define $\hat{\boldsymbol{v}} : \hat{Q} \to \mathbb{R}^n$ by the Piola transform $\hat{\boldsymbol{v}}(\hat{x}) = \operatorname{adj}(B)\boldsymbol{v}(F(\hat{x})) \in \mathbf{\Omega}_3^-(\hat{Q})$, where $\operatorname{adj}(B) =$

$\det(B)B^{-1}$ denotes the adjugate matrix of B. We then have $\widehat{\operatorname{div}}\,\hat{\boldsymbol{v}}(\hat{x}) = \det(B)\operatorname{div}\boldsymbol{v}(x)$ with $x = F(\hat{x})$ [17]. Therefore by Lemma 3.5, scaling arguments and the equivalence of norms in a finite dimensional setting, we obtain

$$\|\boldsymbol{v}\|_{H^1(Q)} \le Ch_Q^{-n/2}\|\hat{\boldsymbol{v}}\|_{H^1(\hat{Q})} \le Ch_Q^{-n/2}\|\widehat{\operatorname{div}}\,\hat{\boldsymbol{v}}\|_{L^2(\hat{Q})} \le C\|\operatorname{div}\boldsymbol{v}\|_{L^2(Q)}.$$

□

4. The Global Stokes Finite Element Spaces and Stability Properties

4.1. Global Finite Element Spaces without Imposed Boundary Conditions. Lemmas 3.2 and 3.4 induce the following global finite element spaces without boundary conditions:

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) :\ \boldsymbol{v}|_Q \in \boldsymbol{\mathfrak{Q}}_3^-(Q)\ \forall Q \in \mathcal{T}_h,\ \partial v^{(i)}/\partial x_i \text{ is continuous across } n-2 \text{ dimensional faces}\},$$

$$W_h := \{q \in L^2(\Omega) :\ q|_Q \in \mathfrak{Q}_2(Q)\ \forall Q \in \mathcal{T}_h,\ q \text{ is continuous across } n-2 \text{ dimensional faces}\}.$$

The goal of this section is to derive the necessary inf–sup condition needed for the well–posedness of the discrete Stokes problem. First we establish a preliminary result.

Lemma 4.1. *For any $q \in W_h$, there exists $\boldsymbol{v}_1 \in \boldsymbol{V}_h$ such that the restriction of $(q - \operatorname{div}\boldsymbol{v}_1)$ to Q is in $\mathring{\mathfrak{Q}}_2(Q)$ for all $Q \in \mathcal{T}_h$. Moreover, $\|\boldsymbol{v}_1\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$.*

Proof. Let $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ satisfy $\operatorname{div}\boldsymbol{w} = q$ and $\|\boldsymbol{w}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$ [13], and denote by $\boldsymbol{w}_h$ the Scott-Zhang interpolant of $\boldsymbol{w}_h$ with $\boldsymbol{w}_h|_Q \in \boldsymbol{\mathfrak{Q}}_1(Q) := [\mathfrak{Q}_1(Q)]^n$ [23]. The error of the interpolant satisfies

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{H^m(Q)} \le Ch_Q^{2-m}\|\boldsymbol{w}\|_{H^1(\omega_Q)} \quad m = 0, 1,$$

where ω_Q denotes the patch of elements that touch Q.

We then uniquely define the function $\boldsymbol{v}_1 \in \boldsymbol{V}_h$ by the following iterative process. On vertices, we specify the conditions $(1 \le i \le n)$

$$\boldsymbol{v}_1(S) = \boldsymbol{w}_h(S), \qquad \frac{\partial v_1^{(i)}}{\partial x_i}(S) = \frac{1}{n}q(S) \qquad S \in \square_0. \tag{4.1a}$$

Now suppose that the values of

$$\int_S \boldsymbol{v}_1 \cdot \boldsymbol{n}_S^{(j)}, \quad \int_S \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{n}_S^{(j)}} \cdot \boldsymbol{n}_S^{(j)} \qquad S \in \square_s,\ j = 1, 2, \ldots, n-s$$

have been specified for $s = 0, 1, \ldots, m$ for some $0 \le m \le n-3$. By Lemma 3.1 and Remark 3.3, this implies that the values of $\boldsymbol{v}$ are well–defined on $S \in \square_m$, and therefore $\int_S \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{t}_S^{(i)}} \cdot \boldsymbol{t}_S^{(i)}$ $(1 \le i \le m+1)$ is well–defined on $S \in \square_{m+1}$ (by the Fundamental Theorem of Calculus). We then construct $\boldsymbol{v}_1$ such that

$$\int_S \boldsymbol{v}_1 \cdot \boldsymbol{n}_S^{(j)} = \int_S \boldsymbol{w}_h \cdot \boldsymbol{n}_S^{(j)}, \qquad \int_S \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{n}_S^{(j)}} \cdot \boldsymbol{n}_S^{(j)} = \frac{1}{n-m-1}\int_S \Big(q - \sum_{i=1}^{m+1} \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{t}_S^{(i)}} \cdot \boldsymbol{t}_S^{(i)}\Big) \qquad S \in \square_{m+1},$$

for $j = 1, 2, \ldots, n-m-1$. Continuing this process up to $m = n-3$, we conclude from Lemma 3.1, Remark 3.3, and Lemma A.1 that

$$\int_S \boldsymbol{v}_1 \cdot \boldsymbol{n}_S^{(j)} = \int_S \boldsymbol{w}_h \cdot \boldsymbol{n}_S^{(j)}, \qquad \int_S \operatorname{div}\boldsymbol{v}_1 = \int_S q \qquad S \in \square_s,\ 0 \le s \le n-2. \tag{4.1b}$$

Finally, the last set of conditions imposed on $\boldsymbol{v}_1$ are

$$\int_S \boldsymbol{v}_1 \cdot \boldsymbol{n}_S = \int_S \boldsymbol{w} \cdot \boldsymbol{n}_S \qquad S \in \square_{n-1}, \tag{4.1c}$$

$$\int_Q \boldsymbol{v}_1 \cdot \boldsymbol{\kappa} = \int_Q \boldsymbol{w}_h \cdot \boldsymbol{\kappa} \qquad \boldsymbol{\kappa} \in \boldsymbol{\mathfrak{Q}}_1^-(Q),\ Q \in \mathcal{T}_h. \tag{4.1d}$$

By Lemma 3.2 and Remark 3.3, this construction uniquely defines $\boldsymbol{v}_1 \in \boldsymbol{V}_h$. Furthermore, the second identity in (4.1b) implies $\operatorname{div} \boldsymbol{v}_1 = q$ on $(n-2)$–dimensional faces (cf. Lemma 2.1). By Stokes theorem and (4.1c) we also have

$$\int_Q \operatorname{div} \boldsymbol{v}_1 = \int_Q \operatorname{div} \boldsymbol{w} = \int_Q q \qquad \forall Q \in \mathcal{T}_h.$$

Consequently, $(q - \operatorname{div} \boldsymbol{v}_1)|_Q \in \mathring{\mathcal{Q}}_2(Q)$ for all $Q \in \mathcal{T}_h$.

It remains to show the stability estimate $\|\boldsymbol{v}_1\|_{H^1(\Omega)} \le \|q\|_{L^2(\Omega)}$. Since (3.3c),(3.4) forms a unisolvent set over $\boldsymbol{\mathcal{Q}}_3^-(Q)$, and since $\boldsymbol{v}_1 - \boldsymbol{w}_h|_Q \in \boldsymbol{\mathcal{Q}}_3^-(Q)$, we have by scaling

$$\begin{aligned}
\|\boldsymbol{v}_1 - \boldsymbol{w}_h\|_{H^1(Q)}^2 &\approx \sum_{s=0}^{n-2} \sum_{S\in\square_s(Q)} \sum_{j=1}^{n-s} h_Q^{n-2s} \Big| \int_S \Big(\frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{n}_S^{(j)}} \cdot \boldsymbol{n}_S^{(j)} - \frac{\partial \boldsymbol{w}_h}{\partial \boldsymbol{n}_S^{(j)}} \cdot \boldsymbol{n}_S^{(j)} \Big) \Big|^2 \\
&\quad + \sum_{S\in\square_{n-1}(Q)} h_Q^{-n} \Big| \int_S (\boldsymbol{v}_1 - \boldsymbol{w}_h)\cdot \boldsymbol{n}_S \Big|^2 \\
&\le \sum_{s=0}^{n} \sum_{S\in\square_s(Q)} h_Q^{n-2s} \Big| \int_S q \Big|^2 + \sum_{s=0}^{n-2} \sum_{S\in\square_s(Q)} \sum_{j=1}^{n-s} h_Q^{n-2s} \Big| \int_S \frac{\partial \boldsymbol{w}_h}{\partial \boldsymbol{n}_S^{(j)}} \cdot \boldsymbol{n}_S^{(j)} \Big|^2 \\
&\quad + \sum_{s=0}^{n-2} \sum_{i=1}^{s} h_Q^{n-2s} \Big| \int_S \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{t}_S^{(i)}} \cdot \boldsymbol{t}_S^{(i)} \Big|^2 \\
&\quad + \sum_{S\in\square_{n-1}(Q)} h_Q^{-n} \Big| \int_S (\boldsymbol{w} - \boldsymbol{w}_h)\cdot \boldsymbol{n}_S \Big|^2.
\end{aligned} \tag{4.2}$$

For a face $S \in \square_s(Q)$ $(1 \le s \le n-2)$ and unit tangent vector $\boldsymbol{t}_S^{(i)}$ $(1 \le i \le s)$, let $S_1, S_2 \in \square_{s-1}(Q)$ be the (unique) $(s-1)$–dimensional faces such that $\boldsymbol{t}_S^{(i)} = \pm \boldsymbol{n}_{S_1}^{(j_1)}$ and $\boldsymbol{t}_S^{(i)} = \pm \boldsymbol{n}_{S_2}^{(j_2)}$ for some $1 \le j_1, j_2 \le (n-s+1)$. Then by the Fundamental Theorem of Calculus and by (4.1b), we have

$$\begin{aligned}
\int_S \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{t}_S^{(i)}} \cdot \boldsymbol{t}_S^{(i)} &= \pm \int_{S_1} \boldsymbol{v}_1 \cdot \boldsymbol{n}_{S_1}^{(j_1)} \pm \int_{S_2} \boldsymbol{v}_1 \cdot \boldsymbol{n}_{S_2}^{(j_2)} \\
&= \pm \int_{S_1} \boldsymbol{w}_h \cdot \boldsymbol{n}_{S_1}^{(j_1)} \pm \int_{S_2} \boldsymbol{w}_h \cdot \boldsymbol{n}_{S_2}^{(j_2)} = \int_S \frac{\partial \boldsymbol{w}_h}{\partial \boldsymbol{t}_S^{(i)}} \cdot \boldsymbol{t}_S^{(i)}.
\end{aligned}$$

Applying this identity to (4.2) and using scaling arguments yields

$$\begin{aligned}
\|\boldsymbol{v}_1 - \boldsymbol{w}_h\|_{H^1(Q)}^2 &\le \sum_{s=0}^{n} \sum_{S\in\square_s(Q)} h_Q^{n-2s} \Big| \int_S q \Big|^2 + \sum_{s=0}^{n-2} \sum_{S\in\square_s(Q)} \sum_{j=1}^{n-s} h_Q^{n-2s} \Big| \int_S \frac{\partial \boldsymbol{w}_h}{\partial \boldsymbol{n}_S^{(j)}} \cdot \boldsymbol{n}_S^{(j)} \Big|^2 \\
&\quad + \sum_{s=0}^{n-2} \sum_{i=1}^{s} h_Q^{n-2s} \Big| \int_S \frac{\partial \boldsymbol{w}_h}{\partial \boldsymbol{t}_S^{(i)}} \cdot \boldsymbol{t}_S^{(i)} \Big|^2 + \sum_{S\in\square_{n-1}(Q)} h_Q^{-n} \Big| \int_S (\boldsymbol{w} - \boldsymbol{w}_h)\cdot \boldsymbol{n}_S \Big|^2 \\
&\le C\big[\|q\|_{L^2(Q)}^2 + \|\boldsymbol{w}_h\|_{H^1(Q)}^2 + h_Q^{-1}\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L^2(\partial Q)}^2\big] \le C\|\boldsymbol{w}\|_{H^1(\omega_Q)}^2.
\end{aligned} \tag{4.3}$$

Summing over $Q \in \mathcal{T}_h$, we conclude that $\|\boldsymbol{v}_1\|_{H^1(\Omega)} \le C\big[\|\boldsymbol{w}_h\|_{H^1(\Omega)} + \|\boldsymbol{w}\|_{H^1(\Omega)}\big] \le C\|\boldsymbol{w}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. □

Theorem 4.2. *For any $q \in W_h$ there exists $\boldsymbol{v} \in \boldsymbol{V}_h$ such that $\operatorname{div} \boldsymbol{v} = q$ and $\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. Consequently, the inf–sup condition*

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_h\setminus\{0\}} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v}) q}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \ge C\|q\|_{L^2(\Omega)} \quad \forall q \in W_h$$

is satisfied.

Proof. For given $q \in W_h$, Lemma 4.1 guarantees the existence of $\boldsymbol{v}_1 \in \boldsymbol{V}_h$ satisfying $(q - \operatorname{div} \boldsymbol{v}_1)|_Q \in \mathring{\mathcal{Q}}_2(Q)$ for all $Q \in \mathcal{T}_h$ and $\|\boldsymbol{v}_1\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. By Theorem 3.7, for each $Q \in \mathcal{T}_h$ there exists $\boldsymbol{v}_{2,Q} \in \mathring{\boldsymbol{\mathfrak{Q}}}_3^-$ such that $\operatorname{div} \boldsymbol{v}_{2,Q} = q - \operatorname{div} \boldsymbol{v}_1$ and $\|\boldsymbol{v}_{2,Q}\|_{H^1(Q)} \le C\|q - \operatorname{div} \boldsymbol{v}_1\|_{L^2(Q)}$. Define $\boldsymbol{v}_2$ such that $\boldsymbol{v}_2|_Q = \boldsymbol{v}_{2,Q}$ and set $\boldsymbol{v} := \boldsymbol{v}_1 + \boldsymbol{v}_2$. Note that the condition $\boldsymbol{v}_2|_Q \in \boldsymbol{\mathfrak{Q}}_3^-(Q) \cap \boldsymbol{H}_0^1(Q)$ for all $Q \in \mathcal{T}_h$ implies $\nabla \boldsymbol{v}_2|_S = 0$ on all $S \in \square_s$ with $0 \le s \le n-2$. Therefore, $\boldsymbol{v} \in V_h$ with $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{v}_1 + \operatorname{div} \boldsymbol{v}_2 = q$, and

$$\|\boldsymbol{v}\|_{H^1(\Omega)} \le \|\boldsymbol{v}_1\|_{H^1(\Omega)} + \|\boldsymbol{v}_2\|_{H^1(\Omega)} \le C(\|\boldsymbol{v}_1\|_{H^1(\Omega)} + \|q - \operatorname{div} \boldsymbol{v}_1\|_{L^2(\Omega)}) \le C\|q\|_{L^2(\Omega)}.$$

□

4.2. **Global Finite Element Spaces with Imposed Boundary Conditions.** As pointed out in [11], imposing boundary conditions of finite element spaces while preserving the surjectivity of the divergence is a non–trivial issue. For example, if $\boldsymbol{v}$ is a globally continuous function on Ω and vanishes on $\partial\Omega$, then the derivatives of $\boldsymbol{v}$ vanish at corners ($n = 2$) and edges ($n = 3$) of $\partial\Omega$. Consequently, the divergence operator is not surjective from $\boldsymbol{V}_h \cap \boldsymbol{H}_0^1(\Omega)$ to $W_h \cap L_0^2(\Omega)$, and therefore the inf–sup condition is lost. More precisely, if we denote by $\square_s^S$ the set of s–dimensional singular faces of $\partial\Omega$, then the gradient of a function $\boldsymbol{v} \in \boldsymbol{V}_h \cap \boldsymbol{H}_0^1(\Omega)$ vanishes on $\partial\Omega_s$ for all $0 \le s \le n-2$. Here, an s–dimensional singular face of $\partial\Omega$ is a hyperplane of codimension $n-s$ that is the intersection of $n-s$ non–parallel $(n-1)$ dimensional hyperplanes of $\partial\Omega$.

On simplicial meshes, this issue may be alleviated by imposing mesh and regularity conditions locally at the boundary [11, Section 3.1]. On cubic meshes, such procedures are not applicable. For example, in two dimensions, there will always be elements in $\mathcal{T}_h$ that have at least two boundary edges. Instead, we shall impose the weaker boundary condition $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ in the finite element space and impose the tangental boundary conditions weakly in the finite element method via Nitsche's method [20]. Thus, we consider the finite element spaces

$$\begin{aligned} \mathring{\boldsymbol{V}}_h &= \{\boldsymbol{v} \in \boldsymbol{V}_h : \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\}, \\ \mathring{W}_h &= W_h \cap L_0^2(\Omega). \end{aligned}$$

Let $\square_s^I$ be the set of (open) s–dimensional faces such that the intersection of a face in $\square_s^I$ with $\partial\Omega$ is empty, and let $\square_s^B = \square_s \backslash \square_s^I$ be the set of boundary s–dimensional faces. We then define the discrete H^1–type norm

$$\|\boldsymbol{v}\|_h^2 = \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 + \sum_{S \in \square_{n-1}^B} \Big(\frac{1}{h_S}\|\boldsymbol{v}\|_{L^2(S)}^2 + h_S\|\partial \boldsymbol{v}/\partial \boldsymbol{n}_S\|_{L^2(S)}^2\Big). \tag{4.4}$$

We now prove the analogue of Theorem 4.2 with boundary conditions.

Theorem 4.3. *The inf-sup condition*

$$\sup_{\boldsymbol{v} \in \mathring{\boldsymbol{V}}_h \backslash \{0\}} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v}) q}{\|\boldsymbol{v}\|_h} \ge C\|q\|_{L^2(\Omega)} \quad \forall q \in \mathring{W}_h \tag{4.5}$$

is satisfied for a constant $C > 0$ independent of h.

Proof. The proof is near identical to the proof of Theorem 4.2. Namely, for given $q \in \mathring{W}_h$, we use the degrees of freedom of $\boldsymbol{V}_h$ to construct a $\boldsymbol{v} \in \mathring{\boldsymbol{V}}_h$ such that $\operatorname{div} \boldsymbol{v} = q$ and $\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$ with $C > 0$ independent of h. This result will imply the inf–sup condition (4.5). We split up the proof into four steps.

Step 1: For $q \in \mathring{W}_h$ let $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$ satisfy $\operatorname{div} \boldsymbol{w} = q$ and $\|\boldsymbol{w}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$ [13]. Set $\boldsymbol{w}_h$ to be the Scott–Zhang interpolant of $\boldsymbol{w}$ such that $\boldsymbol{w}_h|_Q \in \boldsymbol{Q}_1(Q)$ on each $Q \in \mathcal{T}_h$. Note that $\boldsymbol{w}_h \in \boldsymbol{H}_0^1(\Omega)$; in particular, $\boldsymbol{w}_h|_S = 0$ for all boundary faces $S \in \square_s^B$ $(0 \le s \le n-1)$. We then construct $\boldsymbol{v}_1 \in \boldsymbol{V}_h$ by the exact same procedure as in the proof of Lemma 4.1, i.e., $\boldsymbol{v}_1$ is uniquely determined by conditions (4.1). This construction then leads to the property $(q - \operatorname{div} \boldsymbol{v}_1)|_Q \in \mathring{\mathcal{Q}}_2(Q)$ for all $Q \in \mathcal{T}_h$ and $\|\boldsymbol{v}_1\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$.

Step 2: We now establish that $\boldsymbol{v}_1 \in \mathring{\boldsymbol{V}}_h$. Let $S \in \square^B_{n-1}$, and let $\square_s(S)$ with $0 \le s \le n-1$ denote the set of s–dimensional sub–faces of S. Then for a sub–face $S' \in \square_s(S)$ we have $S' \in \square^B_s$. Moreover, the outward unit normal of S is an outward unit normal of S' (up to sign); i.e., $\boldsymbol{n}_S = \pm \boldsymbol{n}^{(j)}_{S'}$ for some $j \in \{1, 2, \ldots, n-s\}$. Therefore by (4.1) we have

$$\int_{S'} \boldsymbol{v}_1 \cdot \boldsymbol{n}_S = \int_{S'} \boldsymbol{w}_h \cdot \boldsymbol{n}_S = 0 \qquad \forall S' \in \square_s(S),\ 0 \le s \le n-1.$$

Thus, since $\boldsymbol{v}_1 \cdot \boldsymbol{n}_S \in \mathcal{Q}_2(S)$ on S, we conclude that $\boldsymbol{v}_1 \cdot \boldsymbol{n}_S \equiv 0$ by Lemma 2.1. Therefore $\boldsymbol{v}_1 \in \mathring{\boldsymbol{V}}_h$.

Step 3: In this step we prove the stability estimate $\|\boldsymbol{v}_1\|_h \le C\|q\|_{L^2(\Omega)}$. For $Q \in \mathcal{T}_h$, set $\hat{\boldsymbol{v}}(\hat{x}) = \boldsymbol{v}(x)$, where $x = F(\hat{x})$ and we recall that $F : \hat{Q} \to Q$ denotes the affine transformation between the reference element $\hat{Q}$ and Q. We then have by scaling and the equivalence of norms in a finite–dimensional setting,

$$\begin{aligned}
&\|\nabla(\boldsymbol{v}_1 - \boldsymbol{w}_h)\|^2_{L^2(Q)} + \sum_{S \in \square_{n-1}(Q) \cap \square^B_{n-1}} \frac{1}{h_S} \|\boldsymbol{v}_1 - \boldsymbol{w}_h\|_{L^2(S)} \\
&\qquad \le C h_Q^{n-2} \Big[\|\hat{\nabla}(\hat{\boldsymbol{v}}_1 - \hat{\boldsymbol{w}}_h)\|^2_{L^2(Q)} + \sum_{\hat{S} \in \square_{n-1}(\hat{Q})} \|\hat{\boldsymbol{v}}_1 - \hat{\boldsymbol{w}}_h\|^2_{L^2(\hat{S})} \Big] \\
&\qquad \le C h_Q^{n-2} \|\hat{\boldsymbol{v}}_1 - \hat{\boldsymbol{w}}_h\|^2_{H^1(\hat{Q})} \le C\|\boldsymbol{v}_1 - \boldsymbol{w}\|^2_{H^1(Q)}.
\end{aligned}$$

Therefore by (4.3) and since $\boldsymbol{w}_h$ vanishes on $\partial\Omega$, we have

$$\|\nabla \boldsymbol{v}_1\|^2_{L^2(Q)} + \sum_{S \in \square_{n-1}(Q) \cap \square^B_{n-1}} \frac{1}{h_S} \|\boldsymbol{v}_1\|^2_{L^2(S)} \le C\big[\|\boldsymbol{w}\|^2_{H^1(\omega_Q)} + \|\nabla \boldsymbol{w}_h\|^2_{L^2(Q)}\big] \le C\|\boldsymbol{w}\|^2_{H^1(\omega_Q)}.$$

Summing over $Q \in \mathcal{T}_h$ then yields

$$\|\nabla \boldsymbol{v}_1\|^2_{L^2(\Omega)} + \sum_{S \in \square^B_{n-1}} \frac{1}{h_S} \|\boldsymbol{v}_1\|^2_{L^2(S)} \le C\|\boldsymbol{w}\|^2_{H^1(\Omega)} \le C\|q\|^2_{L^2(\Omega)}. \tag{4.6}$$

The estimate $\|\boldsymbol{v}_1\|_h \le C\|q\|_{L^2(\Omega)}$ now from (4.6) and scaling.

Step 4: Let $\boldsymbol{v}_{2,Q} \in \mathring{\boldsymbol{\mathcal{Q}}}^-_3(Q)$ satisfy $\operatorname{div} \boldsymbol{v}_{2,Q}|_Q = (q - \operatorname{div} \boldsymbol{v}_1)|_Q$ and set $\boldsymbol{v}_2 \in \boldsymbol{V}_h$ such that $\boldsymbol{v}_2|_Q = \boldsymbol{v}_{2,Q}|_Q$ on each $Q \in \mathcal{T}_h$ (cf. Theorem 3.7). Note that $\boldsymbol{v}_2 \in \boldsymbol{V}_h \cap \boldsymbol{H}^1_0(\Omega)$ and therefore $\boldsymbol{v}_2 \in \mathring{\boldsymbol{V}}_h$. Finally we set $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 \in \mathring{\boldsymbol{V}}_h$. The properties of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ then infer that $\operatorname{div} \boldsymbol{v} = q$ and $\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. □

5. Reduced Elements with Continuous Pressure Approximations

In this section we define a stable pair of Stokes elements with fewer global degrees of freedom by restricting the range of the divergence operator of $\boldsymbol{V}_h$. To construct the spaces, we denote by $B_s(Q) \subset \mathcal{Q}_2(Q)$ the space spanned by the bubble functions associated with $\square_s(Q)$, i.e.,

$$B_s(Q) = \bigoplus_{S \in \square_s(Q)} \langle b_S \rangle,$$

where $\langle b_S \rangle$ denotes the span of b_S. We then have the following decomposition which follows from Lemma 2.1:

$$\mathcal{Q}_2(Q) = \mathcal{Q}_1(Q) \oplus \bigoplus_{s=1}^{n} B_s(Q).$$

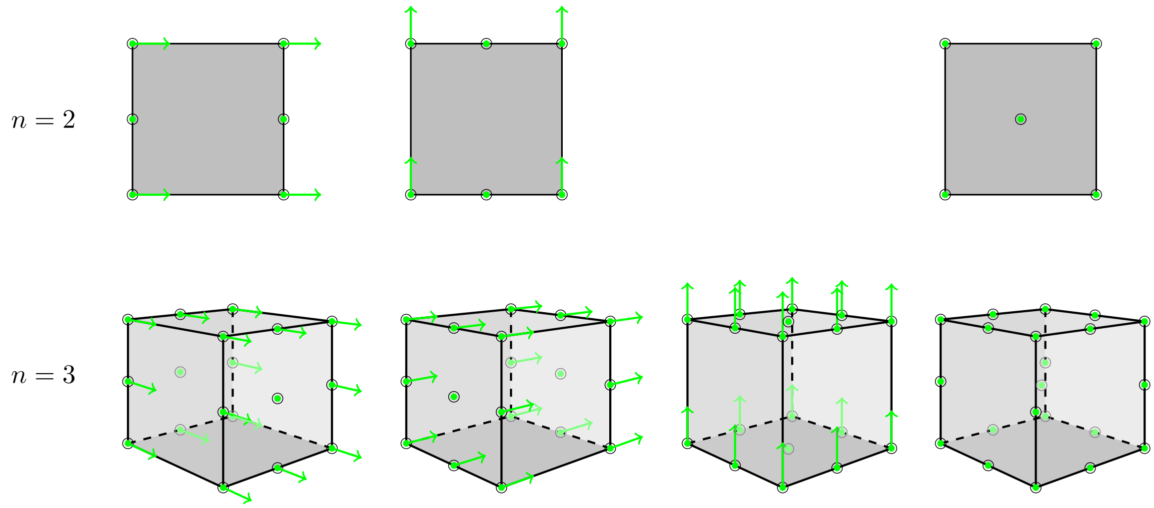

FIGURE 2. Degrees of freedom of the reduced velocity (left) and reduced pressure (right) elements in two and three dimensions.

The local reduced pressure space is then obtained by removing the $(n-1)$–dimensional face bubbles of $\mathcal{Q}_2(Q)$, i.e.,

$$\mathcal{Q}_{2,R}(Q) := \mathcal{Q}_1(Q) \oplus \Big(\bigoplus_{s=1}^{n-2} B_s(Q)\Big) \oplus B_n(Q).$$

For example, in two dimensions $\mathcal{Q}_{2,R}(Q)$ is the space of bilinear polynomials enriched with face bubbles, where as in three dimensions $\mathcal{Q}_{2,R}(Q)$ is the space of trilinear polynomials enriched with edge and volume bubbles. It is easy to see (cf. Lemma 2.1)) that the dimension of $\mathcal{Q}_{2,R}(Q)$ is $3^n - 2n$, and that a function $q \in \mathcal{Q}_{2,R}(Q)$ is uniquely determined by the values

$$\int_S q \qquad S \in \square_s(Q),\ s \in \{0, 1, \dots, n-2, n\}.$$

We note that these degrees of freedom induce a globally continuous finite element space.

The reduced local velocity element consists of functions in $\mathbf{Q}_3^-(Q)$ with divergence in $\mathcal{Q}_{2,R}(Q)$:

$$\mathbf{Q}_{3,R}^-(Q) := \{\boldsymbol{v} \in \mathbf{Q}_3^-(Q) :\ \operatorname{div} \boldsymbol{v} \in \mathcal{Q}_{2,R}(Q)\}.$$

Lemma 5.1. *A function $\boldsymbol{v} \in \mathbf{Q}_{3,R}^-(Q)$ is uniquely determined by the values* (3.3a)–(3.3b)*; see Figure* 2.

Proof. The number of constraints imposed in the definition of $\mathbf{Q}_{3,R}^-(Q)$ is $2n$, and therefore $\dim \mathbf{Q}_{3,R}^-(Q) \ge \dim \mathbf{Q}_3^-(Q) - 2n = 4n3^{n-1} - 2n$; the right–hand side of this inequality being the number of conditions in (3.3a)–(3.3b). Again, we show that if $\boldsymbol{v} \in \mathbf{Q}_{3,R}^-(Q)$ vanishes on (3.3a)–(3.3b), then $\boldsymbol{v} \equiv 0$ to complete the proof.

If $\boldsymbol{v} \in \mathbf{Q}_{3,R}^-(Q) \subset \mathbf{Q}_3^-(Q)$ vanishes on (3.3a)–(3.3b), then $\boldsymbol{v} \in \boldsymbol{H}_0^1(Q)$ by Lemma 3.2. Moreover, the Lemma also shows that $\operatorname{div} \boldsymbol{v}|_S = 0$ for all $S \in \square_s(Q)$ with $0 \le s \le n-2$. By the definition of $\mathcal{Q}_{2,R}(Q)$ and by the properties of the bubble functions, this implies $\operatorname{div} \boldsymbol{v} = \boldsymbol{c} b_Q$ for some $\boldsymbol{c} \in \mathbb{R}^n$. Integration by parts then gives us

$$c \int_Q b_Q = \int_Q \operatorname{div} \boldsymbol{v} = \int_{\partial Q} \boldsymbol{v} \cdot \boldsymbol{n} = 0.$$

Consequently, $\operatorname{div} \boldsymbol{v} = 0$, and therefore by Lemma 3.5, $\boldsymbol{v} \equiv 0$. $\square$

The reduced global velocity and pressure spaces are given respectively by

$$\begin{aligned}\mathring{\boldsymbol{V}}_{R,h} &:= \{\boldsymbol{v} \in \boldsymbol{H}^1(\mathrm{div}\,;\Omega) \cap \boldsymbol{V}_h :\ \boldsymbol{v}|_Q \in \boldsymbol{\mathfrak{Q}}^-_{3,R}(Q),\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\},\\ \mathring{W}_{R,h} &:= \{q \in H^1(\Omega) \cap L^2_0(\Omega) :\ q|_Q \in \mathfrak{Q}_{2,R}(Q)\},\end{aligned}$$

where $\boldsymbol{H}^1(\mathrm{div}\,;\Omega)$ denotes the space of vector–valued $\boldsymbol{H}^1$ function with divergence in $H^1(\Omega)$.

The analogous result given in Theorem 4.3 is provided below for the reduced spaces.

Theorem 5.2. *For every $q \in \mathring{W}_{R,h}$, there exists $\boldsymbol{v} \in \mathring{\boldsymbol{V}}_{R,h}$ such that* $\mathrm{div}\,\boldsymbol{v} = q$ *and* $\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. *Moreover,* $\mathrm{div}\,\mathring{\boldsymbol{V}}_{R,h} = \mathring{W}_{R,h}$.

Proof. By Theorem 4.2, for given $q \in \mathring{W}_{R,h} \subset \mathring{W}_h$, there exists $\boldsymbol{v} \in \mathring{\boldsymbol{V}}_h$ such that $\mathrm{div}\,\boldsymbol{v} = q$ and $\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}$. Since $q \in \mathring{W}_{R,h}$, we have $\boldsymbol{v}|_Q \in \boldsymbol{\mathfrak{Q}}^-_{3,R}(Q)$ on all $Q \in \mathfrak{T}_h$, and $\mathrm{div}\,\boldsymbol{v} \in H^1(\Omega)$. We then conclude that $\boldsymbol{v} \in \mathring{\boldsymbol{V}}_{R,h}$.

The second assertion of the Theorem follows from the first and the inclusion $\mathrm{div}\,\mathring{\boldsymbol{V}}_{R,h} \subseteq \mathring{W}_{R,h}$. □

6. The Stokes Problem and Convergence Analysis

Let $\mathring{\boldsymbol{X}}_h \times \mathring{Y}_h$ denote either the finite element pair $\mathring{\boldsymbol{V}}_h \times \mathring{W}_h$ or the reduced pair $\mathring{\boldsymbol{V}}_{R,h} \times \mathring{W}_{R,h}$. We consider the finite element method: find $(\boldsymbol{u}_h, p_h) \in \mathring{\boldsymbol{X}}_h \times \mathring{Y}_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) - \int_\Omega (\mathrm{div}\,\boldsymbol{v}) p_h = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathring{\boldsymbol{X}}_h, \tag{6.1a}$$

$$\int_\Omega (\mathrm{div}\,\boldsymbol{u}_h) q = 0 \qquad \forall q \in \mathring{Y}_h, \tag{6.1b}$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a_h(\boldsymbol{v}, \boldsymbol{w}) = \nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} - \nu \sum_{S \in \square^B_{n-1}} \int_S \Big(\boldsymbol{v}\cdot\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}_S} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}_S}\cdot\boldsymbol{w} - \frac{\sigma}{h_S}\boldsymbol{v}\cdot\boldsymbol{w}\Big),$$

and $\sigma > 0$ is a positive penalty parameter, independent of h. It is well–known [20] that if σ is taken sufficiently large then $a(\cdot,\cdot)$ is coercive on $\mathring{\boldsymbol{X}}_h$; in particular, $\nu\alpha\|\boldsymbol{v}\|_h^2 \le a_h(\boldsymbol{v},\boldsymbol{v})\ \forall \boldsymbol{v} \in \mathring{\boldsymbol{X}}_h$, where the discrete H^1–type norm $\|\cdot\|_h$ is defined by (4.4). Moreover, the form is continuous on $H^2(\Omega) + \mathring{\boldsymbol{V}}_h$, i.e., $|a(\boldsymbol{v},\boldsymbol{w})| \le C\nu\|\boldsymbol{v}\|_h\|\boldsymbol{w}\|_h$ for all $\boldsymbol{v},\boldsymbol{w} \in H^2(\Omega) + \mathring{\boldsymbol{V}}_h$. Consequently, there exists a unique pair $(\boldsymbol{u}_h, p_h)$ satisfying (6.1) by Theorem 4.3, Theorem 5.2 and standard theory [5, 13]. Furthermore the velocity approximation $\boldsymbol{u}_h$ is independent of the choice of finite element space $\mathring{\boldsymbol{X}}_h = \mathring{\boldsymbol{V}}_h$ or $\mathring{\boldsymbol{X}}_h = \mathring{\boldsymbol{V}}_{R,h}$ since the kernels of these two spaces are the same.

Restricting (6.1) to the kernel $\boldsymbol{Z}_h = \{\boldsymbol{v} \in \mathring{\boldsymbol{X}}_h :\ \mathrm{div}\,\boldsymbol{v} \equiv 0\}$, and using the consistency of the bilinear form $a(\cdot,\cdot)$, we have by Cea's Lemma

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h \le 2 \inf_{\boldsymbol{v} \in \boldsymbol{Z}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_h$$

provided $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ (or more precisely, $\boldsymbol{u} \in \boldsymbol{H}^s$ for some $s > 3/2$). To estimate the approximation properties of $\boldsymbol{Z}_h$, we follow the arguments given in [6, Theorem 12.5.17]. To this end, take an arbitrary $\boldsymbol{v} \in \mathring{\boldsymbol{X}}_h$ and let $\boldsymbol{w} \in \mathring{\boldsymbol{X}}_h$ satisfy $\mathrm{div}\,\boldsymbol{w} = -\mathrm{div}\,\boldsymbol{v} \in \mathring{Y}_h$ and $\|\boldsymbol{w}\|_h \le C\|\mathrm{div}\,\boldsymbol{v}\|_{L^2(\Omega)} = \|\mathrm{div}\,(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(\Omega)} \le C\|\boldsymbol{u}-\boldsymbol{v}\|_h$. We then have $\boldsymbol{w}+\boldsymbol{v} \in \boldsymbol{Z}_h$ and $\|\boldsymbol{u}-(\boldsymbol{w}+\boldsymbol{v})\|_h \le \|\boldsymbol{u}-\boldsymbol{v}\|_h + \|\boldsymbol{w}\|_h \le C\|\boldsymbol{u}-\boldsymbol{v}\|_h$. Consequently,

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_h \le 2 \inf_{\boldsymbol{v} \in \boldsymbol{Z}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_h \le C \inf_{\boldsymbol{v} \in \mathring{\boldsymbol{X}}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_h.$$

If $\mathring{\boldsymbol{X}}_h = \mathring{\boldsymbol{V}}_h$, then $\|\boldsymbol{u} - \boldsymbol{u}_h\|_h \le C \inf_{\boldsymbol{v} \in \mathring{\boldsymbol{V}}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_h \le Ch^s\|\boldsymbol{u}\|_{H^{s+1}(\Omega)}$ $(1 \le s \le 2)$ by standard approximation theory and scaling. Therefore $\|\boldsymbol{u} - \boldsymbol{u}_h\|_h$ satisfies the same estimate in the case $\mathring{\boldsymbol{X}}_h = \mathring{\boldsymbol{V}}_{R,h}$ since $\boldsymbol{u}_h$ is the same approximation.

Denote by $P_h : L^2(\Omega) \to \mathring{Y}_h$ the L^2 projection onto $\mathring{Y}_h$. Since $\operatorname{div} \mathring{\boldsymbol{X}}_h = \mathring{Y}_h$, we have by Theorems 4.3 and 5.2,

$$
\begin{aligned}
C\|p_h - P_h p\|_{L^2(\Omega)} &\le \sup_{\boldsymbol{v}\in \mathring{\boldsymbol{V}}_h\setminus\{0\}} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(p_h - P_h p)}{\|\boldsymbol{v}\|_h} \\
&= \sup_{\boldsymbol{v}\in \mathring{\boldsymbol{V}}_h\setminus\{0\}} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(p_h - p)}{\|\boldsymbol{v}\|_h} = \sup_{\boldsymbol{v}\in \mathring{\boldsymbol{V}}_h\setminus\{0\}} \frac{a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v})}{\|\boldsymbol{v}\|_h} \le C\nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_h.
\end{aligned}
$$

Therefore we have

$$
\|p - p_h\|_{L^2(\Omega)} \le \|p - P_h p\|_{L^2(\Omega)} + C\nu\|\boldsymbol{u}-\boldsymbol{u}_h\|_h \quad \le Ch^s\big(\|p\|_{H^s(\Omega)} + \nu\|\boldsymbol{u}\|_{H^{s+1}(\Omega)}\big) \quad (1 \le s \le 2).
$$

We summarize the results of this section in the following theorem.

Theorem 6.1. *There exists a unique* $(\boldsymbol{u}_h, p_h) \in \mathring{\boldsymbol{X}}_h \times \mathring{Y}_h$ *satisfying* (6.1). *Moreover there holds* $(1 \le s \le 2)$

$$
\|\boldsymbol{u}-\boldsymbol{u}_h\|_h \le Ch^s\|\boldsymbol{u}\|_{H^{s+1}(\Omega)}, \tag{6.2a}
$$

$$
\|p - p_h\|_{L^2(\Omega)} \le Ch^s\big[\|p\|_{H^s(\Omega)} + \nu\|\boldsymbol{u}\|_{H^{s+1}(\Omega)}\big], \tag{6.2b}
$$

where the constant $C > 0$ *is independent of* h, $\boldsymbol{u}$, p *or the viscosity* ν.

Remark 6.2. The velocity error estimate (6.2a) has optimal order of convergence. The pressure error estimate (6.2b) on the other hand is of optimal order provided $\mathring{Y}_h = \mathring{W}_{R,h}$.

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Appendix A. A Calculus Identity

Lemma A.1. *Let $\{\boldsymbol{a}_i\}_{i=1}^n \subset \mathbb{R}^n$ be a set of constant orthonormal (column) vectors. Then there holds*

$$\operatorname{div} \boldsymbol{v} = \sum_{i=1}^n \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}_i} \cdot \boldsymbol{a}_i.$$

Proof. Let A be the orthonormal matrix $A = [\boldsymbol{a}_1|\boldsymbol{a}_2|\cdots|\boldsymbol{a}_n] \in \mathbb{R}^{n\times n}$, and define $\hat{\boldsymbol{v}}(\hat{x}) = A^{-1}\boldsymbol{v}(x) = A^T\boldsymbol{v}(x)$, where $x = A\hat{x}$. We then have $D\boldsymbol{v}(x) = A\hat{D}\hat{\boldsymbol{v}}(\hat{x})A^T$ by the chain rule [5]. Therefore, since the trace is invariant under similarity transforms, and since A is orthonormal, we have

$$\begin{aligned}\sum_{i=1}^n \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{a}_i} \cdot \boldsymbol{a}_i = \sum_{i=1}^n \boldsymbol{a}_i^T (D\boldsymbol{v})\boldsymbol{a}_i = \sum_{i=1}^n (\boldsymbol{a}_i^T A)\hat{D}\hat{\boldsymbol{v}}(\boldsymbol{a}_i^T A)^T = \sum_{i=1}^n \frac{\partial \hat{v}^{(i)}}{\partial \hat{x}_i} \\ = \widehat{\operatorname{div}}\, \hat{\boldsymbol{v}} = \operatorname{tr}(\hat{D}\hat{\boldsymbol{v}}) = \operatorname{tr}(D\boldsymbol{v}) = \operatorname{div} \boldsymbol{v}.\end{aligned}$$

□

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